# Odd-Soliton-Like Solutions for the Variable-Coefficient Variant Boussinesq Model in the Long Gravity Waves

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Under investigation in this paper, with symbolic computation, is a variable-coefficient variant Boussinesq (vcvB) model for the nonlinear and dispersive long gravity waves travelling in two horizontal directions with varying depth. Connection between the vcvB model and a variable-coefficient Broer-Kaup (vcBK) system is revealed under certain constraints. By means of the *N*-fold Darboux transformation for the vcBK system, odd-soliton-like solutions in terms of the Vandermonde-like determinant for the vcvB model are derived. Dynamics of those solutions is analyzed graphically, on the three-parallel solitonic waves, head-on collisions, double structures, and inelastic interactions. It is reported that the shapes of the soliton-like waves and separation distance between them depend on the spectrum parameters and the variable coefficients affect the velocities of the waves. Our results could be helpful in interpreting certain nonlinear wave phenomena in fluid dynamics.

*Key words:* Variable-Coefficient Variant Boussinesq Model; Odd-Soliton-Like Solutions; *N*-Fold Darboux Transformation; Vandermonde-Like Determinant; Symbolic Computation.
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## 1. Introduction

Soliton equations are physically significant [1 – 3]. One of them, the Boussinesq equation [4],

$$\eta_{tt} = \eta_x + (\eta^2)_{xx} + \eta_{xxxx}, \tag{1}$$

has been developed to describe the bidirectional propagation of the small amplitude and long wavelength capillary-gravity waves on the surface of shallow water [5], where $x$ is the scaled space, $t$ is the scaled time, $\eta = \eta(x,t)$ is a sufficiently differentiable function representing the velocity of the water, and the subscripts represent the partial derivatives. Additionally, (1) can be also used to characterize other physical phenomena, such as the nonlinear lattice waves [6], ion sound waves in plasmas [7], vibrations in a nonlinear string [8], and percolation of water in the porous subsurface of a horizontal layer of material [9]. With diverse dispersion and nonlinear effects taken into account, several generalized and variant versions of (1) have also been derived [10, 11].

With the inhomogeneities of media and nonuniformities of boundaries considered, certain variable-coefficient models in fluid dynamics, plasma physics, and optical-fiber communication have attracted considerable attention for their ability to describe the real situations more powerfully than their constant-coefficient counterparts [1 – 3]. In this paper, we shall investigate the variable-coefficient variant Boussinesq (vcvB) model [12],

$$\begin{aligned} &u_t + \alpha_1(t)v_x + \beta_1(t)uu_x + \gamma_1(t)u_{xx} = 0,\\ &v_t + \alpha_2(t)uv_x + \beta_2(t)vu_x + \gamma_2(t)v_{xx} + p(t)u_{xxx} = 0, \end{aligned} \tag{2}$$

where $x$ is the scaled space, $t$ is the scaled time, $u = u(x,t)$ is the field of the horizontal velocity of the water under investigation, and $v = v(x,t)$ denotes the height deviating from the equilibrium position of the water. $p(t)$, $\alpha_j(t)$, $\beta_j(t)$, and $\gamma_j(t)$ $(j = 1,2)$ are some smooth functions of $t$, with $\gamma_1(t)$, $\gamma_2(t)$, and $p(t)$ representing different diffusion strengths. System (2) has been presented for describing the nonlinear and dispersive long gravity waves travelling in two horizontal directions in shallow waters with varying depth [13]. It is observed

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that System (2) includes the following three models arising from the shallow water by the appropriate parameters selection:

- The Broer-Kaup (BK) system [14],

$$u_t = u_{xx} - 2uu_x - 2v_x, \quad v_t = -v_{xx} - 2(uv)_x, \tag{3}$$

characterizes the bidirectional propagation of the long waves in shallow water, where $u = u(x,t)$ is the surface velocity of the water and $v = v(x,t)$ is the wave elevation of the water wave.

- The (1+1)-dimensional dispersive long wave equations [15],

$$p_t + q_x + pp_x = 0, \quad q_t + (pq)_x + \frac{1}{3}p_{xxx} = 0, \tag{4}$$

can be rewritten as the above BK system and admit the bidirectional solitons, where $q$ is the elevation of the water wave and $p$ is the surface velocity of the water along $x$-direction.

- The Whitham-Broer-Kaup model [16, 17],

$$\begin{aligned} &u_t + uu_x + v_x + \beta u_{xx} = 0, \\ &v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0, \end{aligned} \tag{5}$$

is a completely integrable system to describe the dispersive long wave in shallow water. In System (5), $u = u(x,t)$ denotes the horizontal velocity of the water, $v = v(x,t)$ is the height deviating from the equilibrium position of the water, while $\alpha$ and $\beta$ are both real constants representing different diffusion powers.

Auto-Bäcklund transformation and similarity reductions of System (2) have been derived [18]. Painlevé property for System (2) has been reported under some constraints [19], and the $N$-solitonic solutions in terms of the Wronskian determinant has been presented by the Hirota technique [19]. Lax pair and Darboux transformations (DTs) have been constructed for System (2), by which the one- and two-solitonic solutions have been given [20].

Up to now, on the other hand, there have been several ways to obtain the soliton solutions of the soliton equations, such as the inverse scattering transformation [21], Painlevé analysis [22], Bäcklund transformation [23], separated variable method [24], Hirota technique [25], and DT [26]. Among them, the DT based on the Lax pair is a method to generate the soliton solutions of some soliton equations from the trivial seeds [26 – 32]. The key point for constructing the DT is to keep the linear eigenvalue problems associated with the integrable system invariant, so that the new solutions can be obtained from the trivial ones [26]. Especially, the $N$-fold DT, which can be interpreted as a superposition of a single DT, has been applied to certain soliton equations for deriving the multi-soliton solutions [29 – 32]. Compared with that of the single DT, one of the advantages of the $N$-fold DT is that the problem solving of an integrable system is finally reduced to solving a linear system, which is suitable for generating the multi-soliton solutions [29 – 32].

To better understand the dynamics of the water waves in the nonuniform backgrounds and provide useful information for the coastal and civil engineers to apply the nonlinear water wave models in a harbour and coastal design, it is valuable to seek for more solutions of System (2). In this paper, we will focus on the odd-soliton-like solutions in terms of the Vandermonde-like determinant via the $N$-fold DT method and give the analysis on the dynamics for System (2), including the three-parallel solitonic waves, head-on collisions, double structures, and inelastic interactions. To our knowledge, such results have not been reported in the existing literatures as yet.

The outline of this paper, with symbolic computation [1 – 3], will be organized as follows: Connection between System (2) and a variable-coefficient Broer Kaup (vcBK) system will be revealed under certain constraints in Section 2; $N$-fold DT of the vcBK system will be constructed by a gauge transformation in Section 3; as some applications, odd-soliton-like solutions will be presented in terms of the Vandermonde-like determinant and dynamical features for System (2) will be analyzed through figures; Section 4 will be allotted for our conclusion.

## 2. *N*-fold Darboux Transformation (DT)

Applying the following transformation,

$$u = -\frac{1}{2}H, \quad v = \frac{\beta}{2}H_x - G, \tag{6}$$

we can change System (2) into a vcBK system, presented as

$$\begin{aligned} &H_t = a(t)\left(G_x + HH_x - \frac{1}{2}H_{xx}\right), \\ &G_t = a(t)\left[\frac{1}{2}G_{xx} + (GH)_x\right], \end{aligned} \tag{7}$$

under the following constraints:

$$\begin{aligned} \alpha_1(t) &= -\frac{1}{2}a(t), \\ \alpha_2(t) &= \beta_1(t) = \beta_2(t) = 2a(t), \\ \gamma_1(t) &= -\gamma_2(t) = \frac{1}{2}a(t) - \frac{1}{2}\beta a(t), \\ p(t) &= \frac{1}{2}\beta^2 a(t) - \beta a(t), \end{aligned} \tag{8}$$

where $a(t)$ is a smooth function of $t$ and $\beta \neq 0$ is an arbitrary constant.

System (7) is associated with the generalized BK [31] spectral problem,

$$\phi_x = U\phi, \quad \phi_t = V\phi, \tag{9}$$

with

$$U = \begin{pmatrix} \lambda - \frac{1}{2}H & -G \\ 1 & \frac{1}{2}H - \lambda \end{pmatrix}, \quad V = \begin{pmatrix} P & Q \\ R & -P \end{pmatrix}, \tag{10}$$

$$P = \frac{1}{4}a(t)(-H^2 + 4\lambda^2 + H_x), \tag{11}$$

$$Q = -\frac{1}{2}a(t)[G(H+2\lambda) + G_x], \tag{12}$$

$$R = \frac{1}{2}a(t)(H+2\lambda). \tag{13}$$

Compatibility condition $\phi_{xt} = \phi_{tx}$ yields a zero curvature equation,

$$U_t - V_x + [U,V] = 0, \tag{14}$$

which leads to (7).

$N$-fold DTs for the BK system [14] can be seen in [31]. This section discusses the vcBK system, i. e., System (7). Now we introduce a gauge transformation,

$$\overline{\phi} = T\phi, \tag{15}$$

where $T$ is defined by

$$T_x + TU = \overline{U}T, \quad T_t + TV = \overline{V}T. \tag{16}$$

Lax pair (9) can be transformed into

$$\overline{\phi}_x = \overline{U\phi}, \quad \overline{\phi}_t = \overline{V\phi}, \tag{17}$$

where $\overline{U}$ and $\overline{V}$ have the same form as $U$ and $V$, respectively, except replacing $H$ and $G$ with $\overline{H}$ and $\overline{G}$.

Let matrix $T$ in (15) be in the form

$$T = T(\lambda) = \rho \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{18}$$

with

$$\begin{aligned} A &= \lambda^N + \sum_{k=0}^{N-1} \lambda^k A_k, \quad B = \sum_{k=0}^{N-1} \lambda^k B_k, \\ C &= \sum_{k=0}^{N-1} \lambda^k C_k, \quad D = \sum_{k=0}^{N-1} \lambda^k D_k, \end{aligned} \tag{19}$$

where $\rho$, $A_k's$, $B_k's$, $C_k's$, and $D_k's$ $(1 \leq k \leq N-1)$ are the functions of $x$ and $t$. $A_k's$, $B_k's$, $C_k's$, and $D_k's$ can be determined by the following linear algebraic system:

$$\begin{aligned} &\sum_{k=0}^{N-1} \lambda_j^k (A_k + B_k \sigma_j) = -\lambda_j^N, \\ &\sum_{k=0}^{N-1} \lambda_j^k (C_k + D_k \sigma_j) = 0, \end{aligned} \tag{20}$$

with

$$\sigma_j = \frac{\varphi_2(\lambda_j) - r_j \psi_2(\lambda_j)}{\varphi_1(\lambda_j) - r_j \psi_1(\lambda_j)}, \quad (1 \leq j \leq 2N-1), \tag{21}$$

where $\varphi = (\varphi_1, \varphi_2)^{\mathrm{T}}$ and $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ are two basic solutions of the spectral problem (9), and $\lambda_j's$ and $r_j's$ ($\lambda_k \neq \lambda_j$, $r_k \neq r_j$ as $k \neq j$) are some parameters suitably chosen such that the determinant of coefficients for (20) is nonzero.

Hence, if we take

$$B_{N-1} = -\frac{1}{2}G, \quad C_{N-1} = \frac{1}{2}, \tag{22}$$

the rest of $A_k's$, $B_k's$, $C_k's$, and $D_k's$ $(1 \leq k \leq N-1)$ are uniquely determined by (20).

(19) shows that $T(\lambda)$ is a $(2N-1)$th-order polynomial of $\lambda$ and

$$\det T(\lambda_j) = A(\lambda_j)D(\lambda_j) - B(\lambda_j)C(\lambda_j). \tag{23}$$

On the other hand, from (20) we have

$$A(\lambda_j) = -\sigma_j B(\lambda_j), \quad C(\lambda_j) = -\sigma_j D(\lambda_j). \tag{24}$$

Therefore, it holds that

$$\det T(\lambda_j) = 0, \tag{25}$$

which implies that $\lambda_j's$ $(1 \le j \le 2N-1)$ are $2N-1$ roots of $\det T(\lambda)$, i. e.,

$$\det T(\lambda) = \theta \prod_{j=1}^{2N-1} (\lambda - \lambda_j), \tag{26}$$

where $\theta$ is independent of $\lambda$.

We will prove the following theorem:

**Theorem 1:** Let $\rho$ satisfy

$$\rho^2 = \frac{1}{D_{N-1}}, \tag{27}$$

then matrices $\overline{U}$ and $\overline{V}$ have the same forms as $U$ and $V$, respectively, where the transformations from the old potentials into the new are determined by

$$\overline{H} = H + \frac{D_{N-1,x}}{D_{N-1}}, \tag{28}$$

$$\overline{G} = G - 2A_{N-1,x}. \tag{29}$$

**Proof.** Let

$$T^{-1} = \frac{T^*}{\det T}, \tag{30}$$

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}, \tag{31}$$

where $T^*$ denotes the adjoint matrix of $T$. It can be seen that $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are the $2N$th-order polynomials in $\lambda$, while $f_{12}(\lambda)$ and $f_{21}(\lambda)$ are the $(2N-1)$th-order polynomials in $\lambda$. From (9) and (21), we have a Riccati equation

$$\sigma_{jx} = 1 - (2\lambda_j - H)\sigma_j + G\sigma_j^2. \tag{32}$$

Through some direct calculations, all $\lambda_j's(1 \le j \le 2N-1)$ are the roots of $f_{sl}(\lambda)'s$ $(s,l = 1,2)$. Therefore, (31) gives,

$$(T_x + TU)T^* = (\det T)P(\lambda), \tag{33}$$

with

$$P(\lambda) = \begin{pmatrix} f_{11}^{(1)}\lambda + f_{11}^{(0)} & f_{12}^{(0)} \\ f_{21}^{(0)} & f_{22}^{(1)}\lambda + f_{22}^{(0)} \end{pmatrix}, \tag{34}$$

where $f_{sl}^{(j)}{}'s$ $(s,l = 1,2;\ j = 0,1)$ are some undetermined functions independent of $\lambda$. Now (33) can be written as

$$(T_x + TU) = P(\lambda)T. \tag{35}$$

Comparing the coefficients of $\lambda^{N+1}$, $\lambda^N$, and $\lambda^{N-1}$ in (35), we obtain that

$$f_{11}^{(1)} = -f_{22}^{(1)} = 1, \quad f_{21}^{(0)} = 1, \tag{36}$$

$$f_{11}^{(0)} = -f_{22}^{(0)} = -\frac{1}{2}H + \partial_x \ln\rho, \tag{37}$$

$$f_{12}^{(0)} = -G + 2A_{N-1,x}. \tag{38}$$

Substituting (27) into (37) and using $\overline{H}$ in (28), we have

$$f_{11}^{(0)} = -f_{22}^{(0)} = -\frac{1}{2}\overline{H}. \tag{39}$$

Applying $\overline{G}$ in (29), we get

$$f_{12}^{(0)} = -\overline{G}. \tag{40}$$

Therefore, we can obtain $P(\lambda) = \overline{U}$.

Next, we try to prove that $\overline{V}$ has the same form as $V$ under Transformations (28) and (29). Let

$$(T_t + TV)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}, \tag{41}$$

and we see that $g_{11}(\lambda)$ and $g_{22}(\lambda)$ are the $(2N+1)$th-order polynomials in $\lambda$, while $g_{12}(\lambda)$ and $g_{21}(\lambda)$ are the $2N$th-order polynomials in $\lambda$. From (9) and (21), we have a Riccati equation,

$$\begin{aligned}\sigma_{jt} = a(t)\Bigg[&\left(\frac{1}{2}GH + G\lambda_j + \frac{1}{2}G_x\right)\sigma_j^2 \\ &-2\left(-\frac{1}{4}H^2 + \lambda_j^2 + \frac{1}{4}H_x\right)\sigma_j + \frac{1}{2}H + \lambda_j\Bigg].\end{aligned} \tag{42}$$

Through the direct calculations, all $\lambda_j's(1 \le j \le 2N-1)$ are the roots of $g_{sl}(\lambda)'s$ $(s,l = 1,2)$. Therefore, (41) gives

$$(T_t + TV)T^* = (\det T)Q(\lambda), \tag{43}$$

with

$$Q(\lambda) = \begin{pmatrix} g_{11}^{(2)}\lambda^2 + g_{11}^{(1)}\lambda + g_{11}^{(0)} & g_{12}^{(1)}\lambda + g_{12}^{(0)} \\ g_{21}^{(1)}\lambda + g_{21}^{(0)} & g_{22}^{(2)}\lambda^2 + g_{22}^{(1)}\lambda + g_{22}^{(0)} \end{pmatrix}, \tag{44}$$

where $g_{sl}^{(j)}{}'s$ $(s,l = 1,2;\ j = 0,1,2)$ are some undetermined functions independent of $\lambda$. Now (43) can be written as

$$(T_t + TV) = Q(\lambda)T. \tag{45}$$

Comparing the coefficients of $\lambda^{N+2}$, $\lambda^{N+1}$, $\lambda^{N}$, and $\lambda^{N-1}$ in (45), we obtain that

$$g_{11}^{(2)} = -g_{22}^{(2)} = g_{21}^{(1)} = a(t), \quad g_{11}^{(1)} = g_{22}^{(1)} = 0, \tag{46}$$

$$g_{12}^{(1)} = -a(t)\frac{GH + 2GA_{N-1} + 4B_{N-2} + G_x}{2D_{N-1}}, \tag{47}$$

$$g_{21}^{(0)} = -a(t)(A_{N-1} - 2C_{N-2} - D_{N-1}), \tag{48}$$

$$\begin{aligned} g_{11}^{(0)} &= -g_{22}^{(0)} \\ &= \frac{1}{4}\left\{-2g_{12}^{(1)} + 4\frac{\rho_t}{\rho} + a(t)[-H^2 - 2G + H_x]\right\}, \end{aligned} \tag{49}$$

$$\begin{aligned} g_{12}^{(0)} = \frac{1}{8\rho D_{N-1}}\Big\{G\Big[\rho(4g_{11}^{(0)} - a(t)(H^2 + 4A_{N-1}H \\ + 8A_{N-2} - H_x)) - 4\rho_t\Big] - 4\rho\Big[2g_{12}^{(1)}D_{N-2} + G_t \\ + a(t)(4B_{N-3} + A_{N-1}G_x)\Big]\Big\}. \end{aligned} \tag{50}$$

Comparing the coefficients of $\lambda^{N-1}$ and $\lambda^{N-2}$ in (35), we have

$$\begin{aligned} A_{N-1,x} = &-\frac{1}{4D_{N-1}} \\ &\cdot\left[4B_{N-2} + G(H + 2A_{N-1} - 2D_{N-1}) + G_x\right], \end{aligned} \tag{51}$$

$$\begin{aligned} &D_{N-1,x} = \\ &D_{N-1}(-H - 2A_{N-1} + 4C_{N-2} + 2D_{N-1}), \end{aligned} \tag{52}$$

$$\begin{aligned} B_{N-2,x} = &\frac{1}{2D_{N-1}}\Big\{4B_{N-3}D_{N-1} \\ &-2B_{N-2}(2D_{N-2} + HD_{N-1}) \\ &+ G\Big[(-H - 2A_{N-1})D_{N-2} + 2A_{N-2}D_{N-1}\Big] - D_{N-2}G_x\Big\}, \end{aligned} \tag{53}$$

$$\begin{aligned} C_{N-2,x} = &A_{N-2} - 2C_{N-3} - D_{N-2} \\ &+ 2C_{N-2}(-A_{N-1} + 2C_{N-2} + D_{N-1}). \end{aligned} \tag{54}$$

Using $\overline{G}$ in (29) and (51) yields

$$g_{12}^{(1)} = -a(t)\overline{G}. \tag{55}$$

Employing $\overline{H}$ in (28) and (52), we get

$$g_{21}^{(0)} = \frac{1}{2}a(t)\overline{H}. \tag{56}$$

Applying (27), (47) – (49) and then comparing the coefficients of $\lambda^{N-1}$ in (43), we have

$$\begin{aligned} D_{N-1,t} = &\frac{1}{4}a(t)\Big[4B_{N-2} + G(H + 2A_{N-1} - 2D_{N-1}) \\ &+ G_x + 2D_{N-1}(-H^2 + 2D_{N-1}H - 4A_{N-2} \\ &+ 8C_{N-3} + 4D_{N-2} + 4A_{N-1}(A_{N-1} - 2C_{N-2} \\ &- D_{N-1}) + H_x)\Big]. \end{aligned} \tag{57}$$

Noticing (27), (28), (47), (52), (54), and (57), we obtain

$$g_{11}^{(0)} = -\frac{1}{4}a(t)(\overline{H}_x - \overline{H}^2). \tag{58}$$

Using (27) – (29), (47), (49), (51) – (53), and (57), we find that

$$g_{12}^{(0)} = -\frac{a(t)}{2}(\overline{G}_x + \overline{HG}). \tag{59}$$

Therefore, we obtain $Q(\lambda) = \overline{V}$. The proof is completed.

Theorem 1 indicates that the transformations (15), (28), and (29) change the Lax pair (9) into another Lax pair of the same type, i. e., (17). So the two Lax pairs result in System (7). The transformation $(\phi, H, G) \to (\overline{\phi}, \overline{H}, \overline{G})$ is known as a $N$-fold DT of System (7).

## 3. Odd-Soliton-Like Solutions

In this section, we derive the $(2N-1)$-soliton-like solutions for System (2) by applying the aforementioned $N$-fold DT. Substituting $H = 0$ and $G = 1$ into (9), we have two basic solutions

$$\varphi(\lambda_j) = \begin{pmatrix} \cosh\xi_j \\ \lambda_j\cosh\xi_j - c_j\sinh\xi_j \end{pmatrix}, \tag{60}$$

$$\psi(\lambda_j) = \begin{pmatrix} \sinh\xi_j \\ \lambda_j\sinh\xi_j - c_j\cosh\xi_j \end{pmatrix}, \tag{61}$$

with

$$\begin{aligned} &\xi_j = c_j\left[x + \lambda_j\int a(t)\mathrm{d}t\right], \\ &c_j = \sqrt{\lambda_j^2 - 1}, \quad 1 \le j \le 2N - 1. \end{aligned} \tag{62}$$

According to (21), we have

$$\sigma_j = \lambda_j - c_j\frac{\tanh\xi_j - r_j}{1 - r_j\tanh\xi_j}, \quad 1 \le j \le 2N - 1. \tag{63}$$

Let $\lambda_j's$ $(1 \le j \le 2N-1)$ be constants. Then solving System (20), we have

$$A_{N-1} = \frac{\triangle A_{N-1}}{\triangle_1}, \quad D_{N-1} = \frac{\triangle D_{N-1}}{\triangle_2}, \tag{64}$$

with

$$\triangle_1 = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \cdots & \sigma_1\lambda_1^{N-2} & \lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \cdots & \sigma_2\lambda_2^{N-2} & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N-1} & \lambda_{2N-1} & \sigma_{2N-1}\lambda_{2N-1} & \cdots & \sigma_{2N-1}\lambda_{2N-1}^{N-2} & \lambda_{2N-1}^{N-1} \end{vmatrix}, \tag{65}$$

$$\triangle A_{N-1} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \cdots & \sigma_1\lambda_1^{N-2} & -\lambda_1^N + \frac{1}{2}\sigma_1\lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \cdots & \sigma_2\lambda_2^{N-2} & -\lambda_2^N + \frac{1}{2}\sigma_2\lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N-1} & \lambda_{2N-1} & \sigma_{2N-1}\lambda_{2N-1} & \cdots & \sigma_{2N-1}\lambda_{2N-1}^{N-2} & -\lambda_{2N}^N + \frac{1}{2}\sigma_{2N-1}\lambda_{2N-1}^{N-1} \end{vmatrix}, \tag{66}$$

$$\triangle_2 = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \cdots & \sigma_1\lambda_1^{N-2} & \sigma_1\lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \cdots & \sigma_2\lambda_2^{N-2} & \sigma_2\lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N-1} & \lambda_{2N-1} & \sigma_{2N-1}\lambda_{2N-1} & \cdots & \sigma_{2N-1}\lambda_{2N-1}^{N-2} & \sigma_{2N-1}\lambda_{2N-1}^{N-1} \end{vmatrix}, \tag{67}$$

$$\triangle D_{N-1} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \cdots & \sigma_1\lambda_1^{N-2} & -\frac{1}{2}\lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \cdots & \sigma_2\lambda_2^{N-2} & -\frac{1}{2}\lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N-1} & \lambda_{2N-1} & \sigma_{2N-1}\lambda_{2N-1} & \cdots & \sigma_{2N-1}\lambda_{2N-1}^{N-2} & -\frac{1}{2}\lambda_{2N-1}^{N-1} \end{vmatrix}. \tag{68}$$

Hereby, $A_{N-1}$ and $D_{N-1}$ can be expressed in terms of the Vandermonde-like determinant as follows [32–37]:

$$A_{N-1} = \frac{V_{N-1,N}(1;\sigma_j|\lambda_j)}{2(-1)^{N-1}V_{N,N-1}(1;\sigma_j|\lambda_j)} + (-1)^{N+1}\frac{\sum_{k=1}^{2N-1}(-1)^k\lambda_k^N V_{N-1,N-1}[1;\sigma_{l(k)}|\lambda_{l(k)}]}{V_{N,N-1}(1;\sigma_j|\lambda_j)},$$
$$D_{N-1} = (-1)^N\frac{V_{N,N-1}(1;\sigma_j|\lambda_j)}{2V_{N-1,N}(1;\sigma_j|\lambda_j)}, \tag{69}$$

with

$$l(k) = \begin{cases} 2,3,\ldots,2N-1, & k=1, \\ 1,2,\ldots,k-1, \\ k+1,\ldots,2N-1, & 2\le k\le 2N-2, \\ 1,2,\ldots,2N-2, & k=2N-1. \end{cases} \tag{70}$$

Using (6), (28), and (29), we obtain the $(2N-1)$-soliton-like solutions for (2) as follows:

$$u[2N-1] = -\frac{1}{2}\left[\ln(-1)^N\frac{V_{N,N-1}(1;\sigma_j|\lambda_j)}{2V_{N-1,N}(1;\sigma_j|\lambda_j)}\right]_x,$$

$$v[2N-1] = \frac{\beta}{2}\left[\ln(-1)^N\frac{V_{N,N-1}(1;\sigma_j|\lambda_j)}{2V_{N-1,N}(1;\sigma_j|\lambda_j)}\right]_{xx} - 1 + 2\left[\frac{V_{N-1,N}(1;\sigma_j|\lambda_j)}{2(-1)^{N-1}V_{N,N-1}(1;\sigma_j|\lambda_j)}\right]_x + 2\left\{(-1)^{N+1}\frac{\sum_{k=1}^{2N-1}(-1)^k\lambda_k^N V_{N-1,N-1}[1;\sigma_{l(k)}|\lambda_{l(k)}]}{V_{N,N-1}(1;\sigma_j|\lambda_j)}\right\}_x. \tag{71}$$

For $N=2$, $\lambda=\lambda_j's$ $(j=1,2,3)$. Solving (20) leads to

$$A_1 = \frac{\triangle A_1}{\triangle_1}, \quad D_1 = \frac{\triangle D_1}{\triangle_2}, \tag{72}$$

with

$$\triangle_1 = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 \\ 1 & \sigma_2 & \lambda_2 \\ 1 & \sigma_3 & \lambda_3 \end{vmatrix}, \tag{73}$$

$$\triangle A_1 = \begin{vmatrix} 1 & \sigma_1 & -\lambda_1^2 + \frac{1}{2}\lambda_1\sigma_1 \\ 1 & \sigma_2 & -\lambda_2 + \frac{1}{2}\lambda_2\sigma_2 \\ 1 & \sigma_3 & -\lambda_3 + \frac{1}{2}\lambda_3\sigma_3 \end{vmatrix}, \tag{74}$$

$$\triangle_2 = \begin{vmatrix} 1 & \sigma_1 & \lambda_1\sigma_1 \\ 1 & \sigma_2 & \lambda_2\sigma_2 \\ 1 & \sigma_3 & \lambda_3\sigma_3 \end{vmatrix}, \quad (75)$$

$$\triangle D_1 = \begin{vmatrix} 1 & \sigma_1 & -\frac{1}{2}\lambda_1 \\ 1 & \sigma_2 & -\frac{1}{2}\lambda_2 \\ 1 & \sigma_3 & -\frac{1}{2}\lambda_3 \end{vmatrix}. \quad (76)$$

Three-soliton-like solutions of (2) are presented as

$$u[3] = -\frac{1}{2}\frac{D_{1,x}}{D_1}, \quad (77)$$

$$v[3] = \frac{\beta}{2}\left(\frac{D_{1,x}}{D_1}\right)_x - 1 + 2A_{1,x}. \quad (78)$$

**Remark:** The Vandermonde-like determinant is introduced as [34, 35]

$$V_{MN}(a_r;b_r \mid x_r) = \begin{vmatrix} a_1 & a_1x_1 & \cdots & a_1x_1^{M-1} & b_1 & b_1x_1 & \cdots & b_1x_1^{N-1} \\ a_2 & a_2x_2 & \cdots & a_2x_2^{M-1} & b_2 & b_2x_2 & \cdots & b_2x_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{M+N} & a_{M+N}x_{M+N} & \cdots & a_{M+N}x_{M+N}^{M-1} & b_{M+N} & b_{M+N}x_{M+N} & \cdots & b_{M+N}x_{M+N}^{N-1} \end{vmatrix}, \quad (79)$$

where $r = 1,2,\ldots,M+N$. In particular, we denote $V_{MN}(a_r;b_r \mid x_r) = 0$ for $M < 0$ or $N < 0$ and $V_{00}(a_r;b_r \mid x_r) = 1$ for $M = N = 0$. This determinant has some properties and applications in soliton theory [32 – 37].

In order to better understand the dynamics of System (2), we draw some figures to analyze the soliton-like behaviours.

Figures 1 and 2 display the intensity evolution plots of the parallel three-soliton-like solutions for System (2), the traces of which undergo the parabola-type and sine-type oscillation, respectively. Both sets of the three-soliton-like waves propagate stably without any affection to each other and the separation distances among them keep invariant. We can see that the shapes of the soliton-like waves described by $u$ are bell-shaped and those by $v$ are anti-bell-shaped.

Figure 3 (parabola-type) and Figure 4 (sine-type) exhibit the head-on collisions between three soliton-like waves. For example, as shown in Figure 4a, the anti-bell-shaped waves $R_1$ and $R_2$ propagate parallel and do not influence each other, while the direction of the velocity of the bell-shaped wave $R_3$ is reverse to that of $R_1$ (or $R_2$). Head-on collisions continue to arise periodically. Cause of such phenomena is that the algebraic signs of $\lambda_1$ and $\lambda_2$ are opposite to that of $\lambda_3$. Like Figures 1 and 2, it is observed that the amplitudes of the soliton-like waves are always unchangeble while the velocities vary with time due to the effect of $a(t)$.

Compared with Figure 1a and Figure 2a, Figure 5 shows the shape-changing collisions with the parameters $\lambda_j's$ $(j = 1,2,3)$ adjusted. Taking Figure 5b for example, we can see that the soliton-like wave $S_3$ propagates with the invariant amplitude along a sin-type trace while $S_1$ and $S_2$ interact with each other periodically. We observe that Figure 5b plots the opposite trends of the amplitudes of $S_1$ and $S_2$, that is to say, when the amplitude of $S_2$ suppresses, that of $S_1$ will enhance. In fact, contributor to this phenomenon is that the value of $|\lambda_1 - \lambda_2|$ is raised.

By selecting different values of $\lambda_1$, $\lambda_2$, and $\lambda_3$ in Figures 1 and 2, we demonstrate two sorts of the double-humped structures for System (2) in Figure 6. The higher the values of $|\lambda_j|'s$ get, the more apparent such phenomenon becomes.

To our knowledge, the three-parallel solitonic waves (Figs. 1 and 2), head-on collisions (Figs. 3 and 4), inelastic interactions (Fig. 5), and double structures (Fig. 6) for System (2) have not been reported in the published results, as yet, even if authors of [19] have presented the evolution of the two-parallel solitonic waves which do not interact with each other and propagate with a constant separation between them, similar to Figure 1, and presented the fission behaviour that one large-amplitude solitonic wave splits into two small-amplitude solitonic waves. On the other hand, from the figures in [20], one finds that the horizontal velocity $u$ is always the shock wave that is different from those shown in Figures 1 – 6, where $u$ describes the bell-shaped solitonic waves.

From the above analysis, it can be concluded that the shapes of the soliton-like waves and separation distance between them mainly depend on the spectrum parameters $\lambda_j's$ $(1 \le j \le 2N-1)$, and the function $a(t)$ affects the velocities of the soliton-like waves.

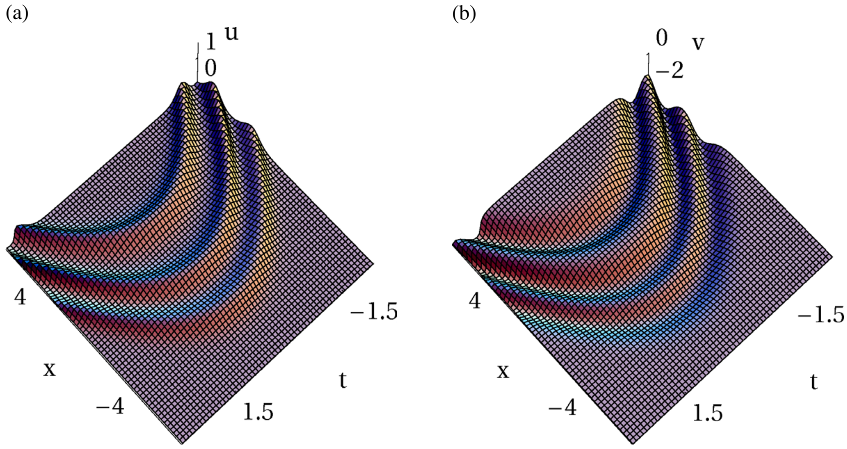


Fig. 1 (colour online). Three-soliton-like solutions via Expressions (77) and (78) with $H=0$, $G=1$, $\lambda_1=-1.5$, $\lambda_2=-1.6$, $\lambda_3=-1.7$, $\beta=1$, $r_1=-0.5$, $r_2=1.5$, $r_3=-0.5$, and $a(t)=t$.

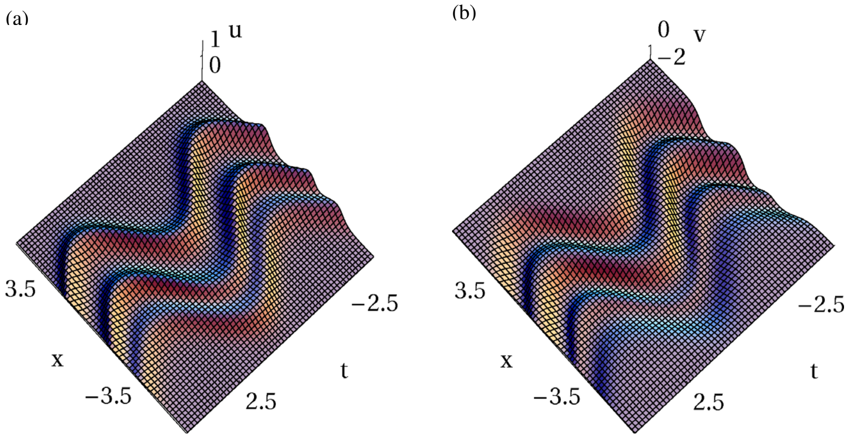


Fig. 2 (colour online). Three-soliton-like solutions via Expressions (77) and (78) with $H=0$, $G=1$, $\lambda_1=-1.5$, $\lambda_2=-1.7$, $\lambda_3=-1.9$, $\beta=1$, $r_1=-0.5$, $r_2=1.2$, $r_3=-0.5$, and $a(t)=\sin(t)$.

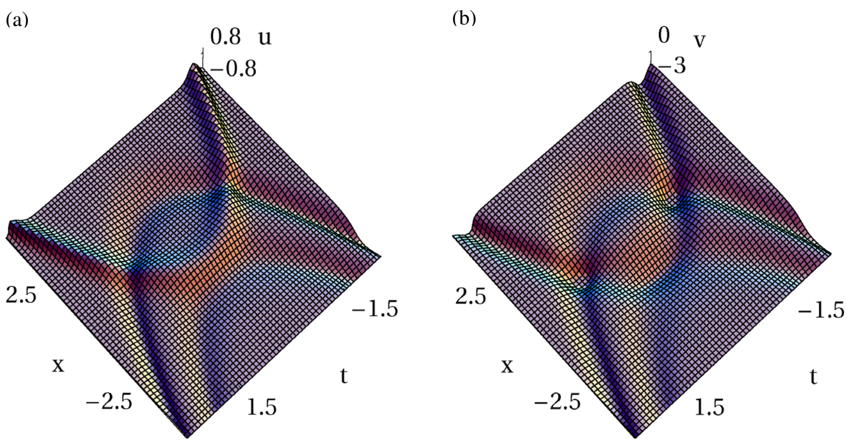


Fig. 3 (colour online). Three-soliton-like solutions via Expressions (77) and (78) with $H=0$, $G=1$, $\lambda_1=1.5$, $\lambda_2=1.6$, $\lambda_3=-1.7$, $\beta=1$, $r_1=-0.5$, $r_2=1.2$, $r_3=-0.5$, and $a(t)=t$.

## 4. Conclusion

In this paper, the main attention has been focused on System (2), which describes the nonlinear and dispersive long gravity waves travelling in two horizontal directions with varying depth. Variable Transformation (6) or the relationship between Systems (2) and (7), has been revealed under the constraints: $\alpha_1(t)=-\frac{1}{2}a(t)$, $\alpha_2(t)=\beta_1(t)=\beta_2(t)=2a(t)$, $\gamma_1(t)=-\gamma_2(t)=\frac{1}{2}a(t)-\frac{1}{2}\beta a(t)$, $p(t)=\frac{1}{2}\beta^2 a(t)-\beta a(t)$.

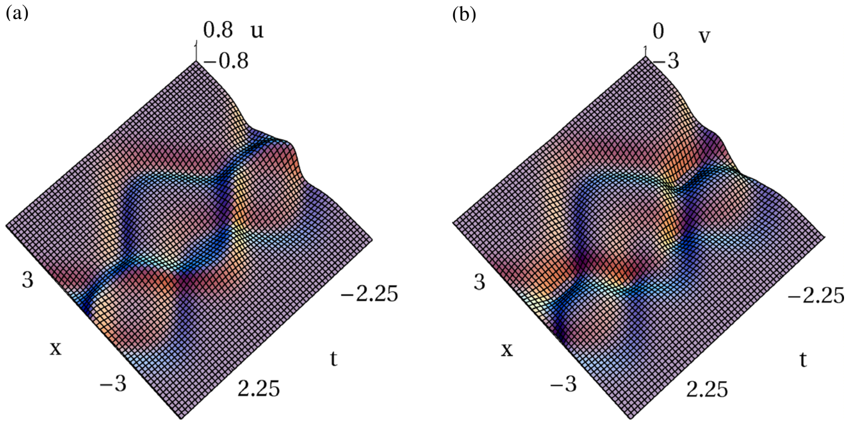


Fig. 4 (colour online). Three-soliton-like solutions via Expressions (77) and (78) with $H = 0$, $G = 1$, $\lambda_1 = 1.5$, $\lambda_2 = 1.6$, $\lambda_3 = -1.7$, $\beta = 1$, $r_1 = -0.5$, $r_2 = 1.2$, $r_3 = -0.5$, and $a(t) = \sin(t)$.

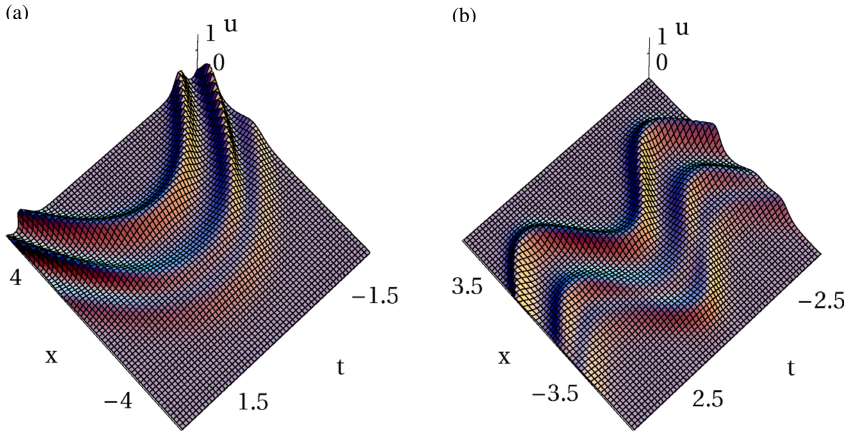


Fig. 5 (colour online). Three-soliton-like solutions via Expression (78). The choice of the parameters: (a) is same as Figure 1 except $\lambda_1 = -1.4$, $\lambda_2 = -1.8$, and $\lambda_3 = -1.9$; (b) is same as Figure 2 except $\lambda_1 = -1.4$, $\lambda_2 = -1.8$, and $\lambda_3 = -1.9$.

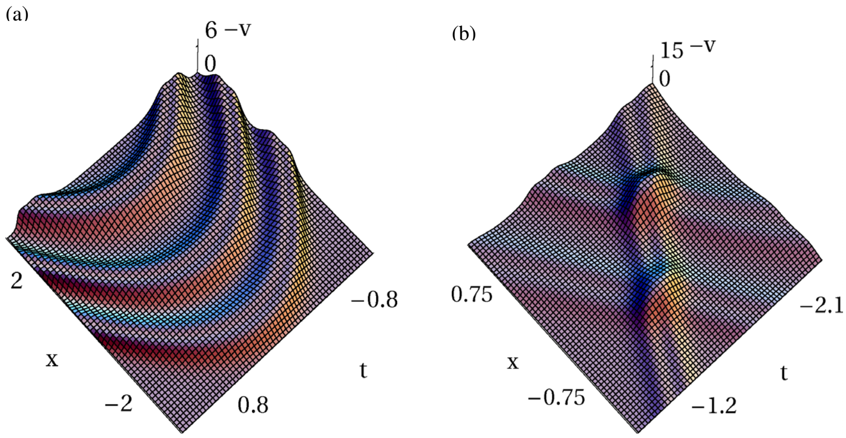


Fig. 6 (colour online). Double-humped structures via Expression (78). The choice of the parameters: (a) is same as Figure 1 except $\lambda_1 = -2.5$, $\lambda_2 = -2.6$, and $\lambda_3 = -2.7$; (b) is same as Figure 4 except $\lambda_1 = 3$, $\lambda_2 = 3.2$, and $\lambda_3 = -3.4$.

$N$-fold DT of Systems (2) [Equations (15), (28), and (29)] have been constructed by a gauge transformation. Multi-soliton-like solutions in terms of the Vandermonde-like determinant for System (2), i. e., Equations (28) and (29), have been presented. Dynamics of the three-soliton-like soltutions have been analyzed graphically, on the three-parallel solitonic waves (Figs. 1 and 2), head-on collisions (Figs. 3 and 4), inelastic interactions (Figs. 5), and double structures (Figs. 6). Relevant issues can be seen in [38, 39].

We have also revealed that the values of the spectral parameters $\lambda_j's$ $(1 \leq j \leq 2N-1)$ have the effect on the interaction between the soliton-like waves as well as the shapes of the soliton-like waves, while the variable coefficients only make the soliton-like waves to change their velocities. Our results could be useful to explain certain nonlinear and dispersive problems in fluid dynamics.

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